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Standard Young tableaux and weight multiplicities of the classical Lie groups

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Abstract. By examining the branching rules for all irreducible representations of the classical groups U(k), SU(k), SO($2k+1$), Sp($2k$) and SO($2k$) on restriction to U(1) × U(1) × ... × U(1), standard Young tableaux are specified for each of these groups. It is shown that these tableaux determine the corresponding characters of the irreducible representations. The rules for constructing these tableaux are derived and in this way the determination of weight multiplicities is reduced to a simple combinatorial exercise. General formulae for such weight multiplicities are given encompassing the most difficult case: namely that of SO($2k$). Illustrative examples are provided, including some yielding the explicit k-dependence of weight multiplicities.

1. Introduction

Recently Stanley (1980), motivated by the work of Patera and Sharp (1979), demonstrated that the standard Young tableaux of SU(k) have an important role to play in determining the character generator of this group. It has also been known for some time (Delaney and Gruber 1969) that such Young tableaux provide a convenient method of determining the weight vectors of a given irreducible representation of SU(k) and the corresponding weight multiplicities.

Surprisingly, much less is known about the generalisation of these ideas to the other classical groups U(k), SO($2k+1$), Sp($2k$) and SO($2k$). The closest approach to the development of standard Young tableaux for these groups came through the work of Gilmore (1970a, b). However, this work did not give all the rules necessary for the construction of such tableaux. It also failed to cover the case of mixed tensor representations of U(k) and did not deal satisfactorily with those pairs of irreducible representations of SO($2k$) arising from the restriction of a single irreducible representation of O($2k$).

This latter problem was also avoided in the most complete analysis of weight multiplicities of the classical groups which is currently available (King and Plunkett 1976). It was only overcome very recently by Wybourne (1982) through the use of difference characters.

Despite this success, achieved without using, either explicitly or implicitly, any standard Young tableaux other than those associated with SU(k), it will be demonstrated here that the generalisation of these tableaux to the other classical groups serves to overcome all the problems of evaluating weight multiplicities, including those associated with SO($2k$). Moreover such tableaux have been shown elsewhere (King 1981, King and El-Sharkaway 1982) to provide a convenient way of determining the character generators of Sp($2k$), SO($2k$) and SO($2k+1$), thus generalising the work of Stanley (1980) on SU(k).

In an earlier paper (King 1975b) the basic ideas were formulated and carried through for U(k) and Sp($2k$) but only in outline for O($2k$) and O($2k+1$), and not at all for SO($2k$). The present paper is intended to remedy the deficiencies of omission in this earlier note.

2. Characters of irreducible representations

The character of each irreducible representation λ_G of a semi-simple Lie group G may be expressed in the form

$$\chi^{\lambda_G}(\boldsymbol{\phi}_G) = \sum_{\boldsymbol{w}} M_{\boldsymbol{w}}^{\lambda_G} \exp(\mathrm{i}\boldsymbol{w} \cdot \boldsymbol{\phi}_G), \tag{2.1}$$

where $\boldsymbol{\phi}_G = (\phi_1, \phi_2, \ldots, \phi_k)$ is a vector whose k real components parametrise the conjugacy classes of G, $\boldsymbol{w}$ is a weight vector and $M_{\boldsymbol{w}}^{\lambda_G}$ is the multiplicity of $\boldsymbol{w}$ in the representation λ_G.

The characters of all the inequivalent irreducible representations of the classical unitary, orthogonal and symplectic groups may be specified by (Black *et al* 1983)

$\{\bar{\mu}; \lambda\}$ with $p+q \leqslant k$	for U(k)
$\{\lambda\}$ with $p \leqslant k-1$	for SU(k)
$[\lambda]$ and $[\Delta; \lambda]$ with $p \leqslant k$	for SO($2k+1$)
$\langle\lambda\rangle$ with $p \leqslant k$	for Sp($2k$)
$[\lambda]$ with $p \leqslant k-1$, $[\lambda]_\pm$ with $p = k$, and $[\Delta; \lambda]_\pm$ with $p \leqslant k$	for SO($2k$)

for all distinct partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_q)$ having p and q non-vanishing parts, respectively. The corresponding highest weight vectors are given in table 1.

In the case of SO($2k$) (King *et al* 1981) the following linear combinations of characters of irreducible representations are required:

$$[\lambda] = [\lambda]_+ + [\lambda]_- \quad \text{and } [\lambda] = [\lambda]_+ - [\lambda]_- \quad \text{with } p = k \tag{2.2a}$$

$$[\Delta; \lambda] = [\Delta; \lambda]_+ + [\Delta; \lambda]_- \quad \text{and } [\Delta; \lambda]'' = [\Delta; \lambda]_+ - [\Delta; \lambda]_- \quad \text{with } p \leqslant k \tag{2.2b}$$

Table 1. Highest weight vectors $\boldsymbol{\lambda}$ of irreducible representations λ_G of the classical groups G.

G	λ_G	$\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_k)$	
U(k)	$\{\bar{\mu}; \lambda\}$	$(\lambda_1, \lambda_2, \ldots, \lambda_p, 0, 0, \ldots, 0, -\mu_q, \ldots, -\mu_2, -\mu_1)$	$p+q \leqslant k$
SU(k)	$\{\lambda\}$	$(\lambda_1, \lambda_2, \ldots, \lambda_p, 0, 0, \ldots, 0)$	$p \leqslant k-1$
SO($2k+1$)	$[\lambda]$	$(\lambda_1, \lambda_2, \ldots, \lambda_p, 0, 0, \ldots, 0)$	$p \leqslant k$
	$[\Delta; \lambda]$	$(\lambda_1+\frac{1}{2}, \lambda_2+\frac{1}{2}, \ldots, \lambda_p+\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$	$p \leqslant k$
Sp($2k$)	$\langle\lambda\rangle$	$(\lambda_1, \lambda_2, \ldots, \lambda_p, 0, 0, \ldots, 0)$	$p \leqslant k$
SO($2k$)	$[\lambda]$	$(\lambda_1, \lambda_2, \ldots, \lambda_p, 0, 0, \ldots, 0)$	$p \leqslant k-1$
	$[\lambda]_\pm$	$(\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \pm\lambda_k)$	$p = k$
	$[\Delta; \lambda]_\pm$	$(\lambda_1+\frac{1}{2}, \lambda_2+\frac{1}{2}, \ldots, \lambda_p+\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \pm\frac{1}{2})$	$p \leqslant k-1$
		$(\lambda_1+\frac{1}{2}, \lambda_2+\frac{1}{2}, \ldots, \lambda_{k-1}+\frac{1}{2}, \pm\lambda_k \pm\frac{1}{2})$	$p = k$

and it is convenient to introduce

$$[\square;\lambda]=[\square;\lambda]_+ +[\square;\lambda]_- \text{ and } [\square;\lambda]''=[\square;\lambda]_+ -[\square;\lambda]_- \qquad \text{with } p\leqslant k \qquad (2.2c)$$

where $\square;\lambda$ denotes the k-part partition $(\lambda_1+1, \lambda_2+1, \ldots, \lambda_p+1, 1, 1, \ldots, 1)$.

Elements of each of the classical groups G which belong to the conjugacy class labelled by $\boldsymbol{\phi}_G$ take the form

$$\begin{bmatrix} e^{i\phi_1} & & & \\ & e^{i\phi_2} & & \\ & & \ddots & \\ & & & e^{i\phi_k} \end{bmatrix} \quad \text{for } U(k)$$

$$\begin{bmatrix} e^{i\phi_1} & & & \\ & e^{i\phi_2} & & \\ & & \ddots & \\ & & & e^{i\phi_k} \end{bmatrix} \quad \text{with } \phi_1+\phi_2+\ldots+\phi_k=0 \text{ for } SU(k)$$

$$\begin{bmatrix} \cos\phi_1 & -\sin\phi_1 & & & & & & & \\ \sin\phi_1 & \cos\phi_1 & & & & & & & \\ & & \cos\phi_2 & -\sin\phi_2 & & & & & \\ & & \sin\phi_2 & \cos\phi_2 & & & & & \\ & & & & \ddots & & & & \\ & & & & & \cos\phi_k & -\sin\phi_k & \\ & & & & & \sin\phi_k & \cos\phi_k & \\ & & & & & & & 1 \end{bmatrix} \quad \text{for } SO(2k+1)$$

$$\begin{bmatrix} e^{-i\phi_1} & & & & & & \\ & e^{i\phi_1} & & & & & \\ & & e^{-i\phi_2} & & & & \\ & & & e^{i\phi_2} & & & \\ & & & & \ddots & & \\ & & & & & e^{-i\phi_k} & \\ & & & & & & e^{i\phi_k} \end{bmatrix} \quad \text{for } Sp\,(2k)$$

$$\begin{bmatrix} \cos\phi_1 & -\sin\phi_1 & & & & & \\ \sin\phi_1 & \cos\phi_1 & & & & & \\ & & \cos\phi_2 & -\sin\phi_2 & & & \\ & & \sin\phi_2 & \cos\phi_2 & & & \\ & & & & \ddots & & \\ & & & & & \cos\phi_k & -\sin\phi_k \\ & & & & & \sin\phi_k & \cos\phi_k \end{bmatrix} \quad \text{for } SO(2k).$$

With this notation the explicit dependence of the characters $\chi^{\lambda_G}(\boldsymbol{\phi}_G)$ upon ϕ_k may be found through a consideration of the branching rules appropriate to the group–

subgroup restrictions:

$$\mathrm{U}(k)\downarrow\mathrm{U}(k-1)\times\mathrm{U}(1)$$
$$\mathrm{SU}(k)\downarrow(\mathrm{U}(k-1)\times\mathrm{U}(1))/\mathrm{U}(1)$$
$$\mathrm{SO}(2k+1)\downarrow\mathrm{SO}(2k)\downarrow\mathrm{SO}(2k-2)\times\mathrm{SO}(2)\approx\mathrm{SO}(2k-2)\times\mathrm{U}(1)$$
$$\mathrm{Sp}(2k)\downarrow\mathrm{Sp}(2k-2)\times\mathrm{Sp}(2)\downarrow\mathrm{Sp}(2k-2)\times\mathrm{U}(1)$$
$$\mathrm{SO}(2k)\downarrow\mathrm{SO}(2k-2)\times\mathrm{SO}(2)\approx\mathrm{SO}(2k-2)\times\mathrm{U}(1).$$

In the second of these the factoring out of U(1) is necessitated by the constraint $\phi_1+\phi_2+\ldots+\phi_k=0$. For this reason it is preferable to identify the characters $\{\lambda\}$ of SU(k) with the corresponding characters $\{\lambda\}=\{\bar{0};\lambda\}$ of U(k) subject to this constraint, and to work with the group–subgroup restriction $\mathrm{U}(k)\downarrow\mathrm{U}(k-1)\times\mathrm{U}(1)$.

If this is done then all the group–subgroup restrictions are of the form $G\downarrow H\times\mathrm{U}(1)$ and yield:

$$\chi^{\lambda_G}(\boldsymbol{\phi}_G)=\sum_{\sigma_H,w_k} B^{\lambda_G}_{\sigma_H\{w_k\}}\chi^{\sigma_H}(\boldsymbol{\phi}_H)\chi^{\{w_k\}}(\phi_k) \tag{2.3}$$

where $\boldsymbol{\phi}_G=(\phi_1,\phi_2,\ldots,\phi_{k-1},\phi_k)$ and $\boldsymbol{\phi}_H=(\phi_1,\phi_2,\ldots,\phi_{k-1})$, whilst σ_H and $\{w_k\}$ label irreducible representations of H and U(1) respectively. Moreover

$$\chi^{\{w_k\}}(\phi_k)=\exp(\mathrm{i}w_k\phi_k) \tag{2.4}$$

so that a knowledge of the branching rule coefficients $B^{\lambda_G}_{\sigma_H\{w_k\}}$ is then sufficient to determine the dependence of $\chi^{\lambda_G}(\boldsymbol{\phi}_G)$ on ϕ_k. Moreover, iterating these branching rules with k replaced by $k-1, k-2, \ldots, 2, 1$ will lead to an expression for $\chi^{\lambda_G}(\boldsymbol{\phi}_G)$ of the required form (2.1).

3. Branching rules and modification rules

The relevant branching rules may be derived in a variety of ways, but the use of Schur function methods (King 1975a, King *et al* 1981) provides a common framework in which to deal with them all. The required results are summarised in table 2. The notation is such that x, y and z are one-part partitions, whilst / and signify Schur function quotients and products, both operations being governed by the Littlewood–Richardson rule (Littlewood 1940, p 94).

These results conform with the branching rules given by Weyl (1931, p 391) for $\mathrm{U}(k)\downarrow\mathrm{U}(k-1)$, by Boerner (1970, pp 267, 269) for $\mathrm{SO}(2k+1)\downarrow\mathrm{SO}(2k)$ and $\mathrm{SO}(2k)\downarrow\mathrm{SO}(2k-1)$, and by Miller (1966) and Hegerfeldt (1967) for $\mathrm{Sp}(2k)\downarrow\mathrm{Sp}(2k-2)$. The additional feature included here is the explicit dependence upon the characters of U(1) which is crucial to our development. The origin of this dependence goes back to the special case of the branching rule (Whippman 1965, King 1975a)

$$\mathrm{U}(p+q)\downarrow\mathrm{U}(p)\times\mathrm{U}(q) \qquad \{\lambda\}\downarrow\sum_{\xi}\{\lambda/\xi\}\times\{\xi\} \tag{3.1}$$

for which $p=k-1$ and $q=1$. In this special case the summation over ξ is confined to the one-part partition $\xi=x$, as indicated in table 2. The corresponding U(1) character is $\{x\}=\exp(\mathrm{i}x\phi_k)$. Although expressed somewhat differently, this result for the branching from U(k) to $\mathrm{U}(k-1)\times\mathrm{U}(1)$ seems to have first been made explicit by

Table 2. Branching rules for $G{\downarrow}H\times\mathrm{U}(1)$.

$\mathrm{U}(k){\downarrow}\mathrm{U}(k-1)\times\mathrm{U}(1)$	$\{\lambda\}{\downarrow}\sum_{x}\{\lambda/x\}\times\{x\}$
	$\{\bar{\mu};\lambda\}{\downarrow}\sum_{x,y}\{\overline{\mu/y};\lambda/x\}\times\{x-y\}$
$\mathrm{SO}(2k+1){\downarrow}\mathrm{SO}(2k)$	$[\lambda]{\downarrow}\sum[\lambda/z]$
	$[\Delta;\lambda]{\downarrow}\sum_{z}[\Delta;\lambda/z]$
$\mathrm{Sp}(2k){\downarrow}\mathrm{Sp}(2k-2)\times\mathrm{U}(1)$	$\langle\lambda\rangle{\downarrow}\sum_{x,y}\langle\lambda/x\cdot y\rangle\times\{x-y\}$
$\mathrm{SO}(2k){\downarrow}\mathrm{SO}(2k-2)\times\mathrm{U}(1)$	$[\lambda]{\downarrow}\sum_{x,y}[\lambda/x\cdot y]\times\{x-y\}$
	$[\Delta;\lambda]{\downarrow}\sum_{x,y}[\Delta;\lambda/x\cdot y]\times(\{x-y+\frac{1}{2}\}+\{x-y-\frac{1}{2}\})$
	$[\Delta;\lambda]''{\downarrow}\sum_{x,y}[\Delta;\lambda/x\cdot y]''\times(\{x-y+\frac{1}{2}\}-\{x-y-\frac{1}{2}\})$
	$[\Delta;\lambda]_{\pm}{\downarrow}\sum_{x,y}([\Delta;\lambda/x\cdot y]_{\pm}\times\{x-y+\frac{1}{2}\}$
	$+[\Delta;\lambda/x\cdot y]_{\mp}\times\{x-y-\frac{1}{2}\})$
	$[\Box;\lambda]{\downarrow}\sum_{x,y}[(\Box;\lambda)/x\cdot y]\times\{x-y\}$
	$[\Box;\lambda]''{\downarrow}\sum_{x,y}[\Box;\lambda/x\cdot y]''\times(\{x-y+1\}-\{x-y-1\})$

Zhelobenko (1962) and to have first been used to determine weight multiplicities by Gilmore (1970a).

The result tabulated for $\mathrm{Sp}(2k){\downarrow}\mathrm{Sp}(2k-2)\times\mathrm{U}(1)$ was also given first by Zhelobenko (1962) and subsequently used by Gilmore (1970a). It may be derived most easily through a consideration of the chain

$$\mathrm{Sp}(2k){\uparrow}\mathrm{U}(2k){\downarrow}(2k-1)\times\mathrm{U}(1){\downarrow}\mathrm{U}(2k-2)\times\mathrm{U}(1)\times\mathrm{U}(1){\downarrow}\mathrm{Sp}(2k-2)\times\mathrm{U}(1). \qquad (3.2)$$

Making use of certain infinite series of Schur functions (Littlewood 1940, p 238, King 1975a) this yields the result

$$\langle\lambda\rangle{\uparrow}\{\lambda/A\}{\downarrow}\sum_{x}\{\lambda/Ax\}\times\{x\}{\downarrow}\sum_{x,y}\{\lambda/Axy\}\times\{x\}\times\{\bar{y}\}{\downarrow}\sum_{x,y}\langle\lambda/AxyB\rangle\times\{x-y\}$$

$$=\sum_{x,y}\langle\lambda/xy\rangle\times\{x-y\} \qquad \text{with } xy=x\cdot y \qquad (3.3)$$

as given in table 2. It should be pointed out that the first of the U(1) groups is associated with $\exp(\mathrm{i}\phi_k)$ and the second with $\exp(-\mathrm{i}\phi_k)$. This is the origin of the distinction between $\{x\}=\exp(\mathrm{i}x\phi_k)$ and $\{\bar{y}\}=\exp(-\mathrm{i}y\phi_k)$ whose product yields $\{x-y\}=\exp\mathrm{i}(x-y)\phi_k$.

This derivation may be adapted to deal with the characters of SO($2k$) through a consideration of the chain

$$\mathrm{SO}(2k){\uparrow}\mathrm{U}(2k){\downarrow}\mathrm{U}(2k-1)\times\mathrm{U}(1){\downarrow}\mathrm{U}(2k-2)\times\mathrm{U}(1)\times\mathrm{U}(1){\downarrow}\mathrm{SO}(2k-2)\times\mathrm{U}(1). \qquad (3.4)$$

This yields

$$[\lambda]\uparrow\{\lambda/C\}\downarrow\sum_x\{\lambda/Cx\}\times\{x\}\downarrow\sum_{x,y}\{\lambda/Cxy\}\times\{x\}\times\{\bar{y}\}\downarrow\sum_{x,y}[\lambda/CxyD]\times\{x-y\}$$

$$=\sum_{x,y}[\lambda/xy]\times\{x-y\}. \tag{3.5}$$

To complete the results it is only necessary to note the particular branching rules

$$\mathrm{SO}(2k)\downarrow\mathrm{SO}(2k-2)\times\mathrm{SO}(2)\simeq\mathrm{SO}(2k-2)\times\mathrm{U}(1)$$

$$\Delta\downarrow\Delta\times\Delta=\Delta\times(\{\tfrac{1}{2}\}+\{\tfrac{\bar{1}}{2}\})$$

$$\Delta''\downarrow\Delta''\times\Delta''=\Delta''\times(\{\tfrac{1}{2}\}-\{\tfrac{\bar{1}}{2}\})$$

and

$$\Box''=\Delta\cdot\Delta''\downarrow\Delta\cdot\Delta''\times\Delta\cdot\Delta''=\Box''\times\Box''=\Box''\times(\{1\}-\{\bar{1}\})$$

and to use the identities (King *et al* 1981):

SO(2k)	$[\Delta;\lambda]=\Delta\cdot[\lambda/P]$	$\Delta\cdot[\lambda]=[\Delta;\lambda/Q]$
	$[\Delta;\lambda]''=\Delta''\cdot[\lambda/M]$	$\Delta''\cdot[\lambda]=[\Delta;\lambda/L]''$
	$[\Box;\lambda]''=\Box''\cdot[\lambda/W]$	$\Box''\cdot[\lambda]=[\Box;\lambda/V]''$

along with

$$AB=CD=PQ=LM=WV=1.$$

This technique was outlined elsewhere (King 1982) but has now been made completely explicit by Black and Wybourne (1983) to cover the case of all irreducible representations of SO(2k) branching under the restriction to SO(2k − 2) × U(1).

In applying the branching rules of table 2 it should be stressed that certain non-standard labels σ_H of irreducible representations of H may arise. These may be dealt with by using the modification rules of table 3. These are the appropriate special cases of the complete set of such modification rules described elsewhere (King 1971, 1975a, King *et al* 1981, Wybourne 1982). They are nothing other than identities between characters specified by standard and non-standard labels. They extend the modification rules given by Murnaghan (1938, p 282) and Newell (1951) which were

Table 3. Modification rules.

U(k − 1)	$\{\sigma_1\sigma_2\ldots\sigma_{k-1}\sigma_k\}=0$	if $\sigma_k>0$
	$\{\overline{\tau_1\tau_2\ldots\tau_{k-r}};\sigma_1\sigma_2,\ldots\sigma_r\}=0$	if $\sigma_r>0$ and $\tau_{k-r}>0$
Sp(2k − 2)	$\langle\sigma_1\sigma_2\ldots\sigma_{k-1}\sigma_k\rangle=0$	if $\sigma_k>0$
SO(2k − 2)	$[\sigma_1\sigma_2\ldots\sigma_{k-1}\sigma_k]=0$	if $\sigma_k>2$
	$[\sigma_1\sigma_2\ldots\sigma_{k-1}\sigma_k]=-[\sigma_1\sigma_2\ldots\sigma_{k-1}]$	if $\sigma_k=2$
	$[\sigma_1\sigma_2\ldots\sigma_{k-1}\sigma_k]=0$	if $\sigma_k=1$ and $\sigma_{k-1}>1$
	$[\sigma_1\sigma_2\ldots\sigma_{k-1}\sigma_k]=[\sigma_1\sigma_2\ldots\sigma_{k-2}0]$	if $\sigma_k=\sigma_{k-1}=1$
	$[\Delta;\sigma_1\sigma_2\ldots\sigma_{k-1}\sigma_k]=0$	if $\sigma_k>1$
	$[\Delta;\sigma_1\sigma_2\ldots\sigma_{k-1}\sigma_k]=-[\Delta;\sigma_1\sigma_2\ldots\sigma_{k-1}]$	if $\sigma_k=1$
	$[\Delta;\sigma_1\sigma_2\ldots\sigma_{k-1}\sigma_k]''=0$	if $\sigma_k>1$
	$[\Delta;\sigma_1\sigma_2\ldots\sigma_{k-1}\sigma_k]''=[\Delta;\sigma_1\sigma_2\ldots\sigma_{k-1}]''$	if $\sigma_k=1$
	$[\Box;\sigma_1\sigma_2\ldots\sigma_{k-1}\sigma_k]''=0$	if $\sigma_k>0$

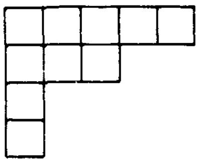
used in precisely this same context by Gilmore (1970a). The extension referred to takes in both the characters $[\Delta, \sigma]''$ and $[\square; \sigma]''$ of SO$(2k-2)$. The use of difference characters serves to fill the gap left by Gilmore (1970a, b) in dealing so successfully with all cases other than the characters of the irreducible representations $[\square; \lambda]_\pm$ of SO$(2k)$.

4. Enumeration using Young diagrams

The enumeration of the terms arising through the use of the branching rules of the preceding section may conveniently be accomplished by making use of Young diagrams or frames.

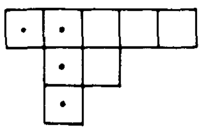
Each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ with $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_p > 0$, $\lambda_1 + \lambda_2 + \ldots + \lambda_p = l$ and λ_i an integer for $i = 1, 2, \ldots, p$ serves to specify a Young diagram, F^λ, consisting of l boxes arranged in p rows of length $\lambda_1, \lambda_2, \ldots, \lambda_p$, left-adjusted to a vertical line. For example

$F^{(531^2)} =$



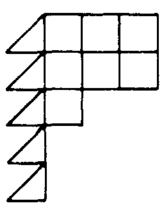
Such Young diagrams may be generalised in several ways. Firstly (Abramsky and King 1970, King 1970) the label $\bar{\mu}; \lambda$, used in the description of mixed tensor representations of U(k), specifies a Young diagram $F^{\bar{\mu};\lambda}$. This is formed by placing the Young diagrams F^μ and F^λ back-to-back in such a way that the boxes in the rows of F^μ and F^λ are right-adjusted and left-adjusted, respectively, to the same vertical line. The boxes of F^μ are distinguished from those of F^λ by being dotted. Thus for example

$F^{(\bar{2}\bar{1}^2;31)} =$



Secondly the label $\Delta;\lambda$, used in the description of spinor representations of SO$(2k+1)$ and SO$(2k)$, may be used to specify a Young diagram $F^{\Delta;\lambda}$ consisting of a column of half boxes of length k adjoined to the left-hand side of F^λ. This differs only marginally from the diagrammatic notation introduced by Gilmore (1970b). Here for example if $k = 5$

$F^{(\Delta;321)} =$



Similarly for consistency the Young diagram, $F^{\square;\lambda}$, is formed by adjoining a column of boxes of length k to the left-hand side of F^{λ}. This yields a conventional Young diagram of course. For example if $k=4$

$$F^{(\square;42)}=F^{(531^2)}.$$

Each Young diagram may be numbered in a variety of ways by inserting entries into the boxes or half boxes of the diagram. Such numberings may be generated by a consideration of the branching rules of table 2.

The key operation in the branching rules is that of Schur function division. This operation is governed by the Littlewood–Richardson rule (Littlewood 1940, p 94). In the special case required here this implies that

$$\lambda/x=\sum_{\sigma}\sigma$$

where the summation is over all partitions σ such that F^{σ} may be obtained from F^{λ} by the removal of x boxes, with no more than one box removed from any one column. If λ and σ are partitions of l and s respectively then $s=l-x$. It is convenient here to keep account of the x removed boxes by inserting in each of them a numerical entry (in contrast to the literal entry used by Littlewood).

To be precise in dealing with the branching rules of table 2 each box, or dotted box, removed from F^{λ}, $F^{\bar{\mu};\lambda}$, $F^{\Delta;\lambda}$ and $F^{\square;\lambda}$ through division by x, y or z should be replaced by an entry k, $\bar{k}$ or 0 respectively. The x entries k, y entries $\bar{k}$ or z entries 0 are to be associated with the U(1) characters $\{x\}$, $\{\bar{y}\}$ or $\{0\}$ respectively.

Similarly each half box removed from the first columns of $F^{\Delta;\lambda}$ in branching from $[\Delta;\lambda]$ or $[\Delta;\lambda]''$ of SO($2k$) to $[\Delta;\sigma]$ or $[\Delta;\sigma]''$ of SO($2k-2$) should be replaced by a half entry k or $\bar{k}$, to be associated with the U(1) characters $\{\frac{1}{2}\}$ or $\{\bar{\frac{1}{2}}\}$, respectively. Finally each box removed from the first column of $F^{\square;\lambda}$ in branching from $[\square;\lambda]''$ of SO($2k$)to $[\square;\sigma]''$ of SO($2k-2$) should be replaced by an entry k or $\bar{k}$, to be associated with the U(1) characters $\{1\}$ or $\{\bar{1}\}$ respectively.

It is clear that in this way the entries k and $\bar{k}$ provide a very convenient way of determining the U(1) character to be associated with each term σ_H arising from λ_G under the restriction of G to $H\times$U(1).

The only difficulty is that occasioned by the need to use modification rules. These rules are given in table 3 and must be used to standardise the representation labels for H.

The modification rules for U($k-1$) are such that the only contributions to the character $\{\lambda\}$ of U(k) are those enumerated by making entries k in the boxes of F^{λ} in such a way that if $p=k$ then each and every box of the kth row contains k.

Similarly the only contributions to the character $\{\bar{\mu};\lambda\}$ of U(k) are those enumerated by making entries k in the boxes of $F^{\bar{\mu};\lambda}$ and entries $\bar{k}$ in the dotted boxes of $F^{\bar{\mu};\lambda}$ in such a way that if $p+q=k$ then either each and every box of the pth row contains k if any dotted box of the qth row is unfilled, or each and every dotted box of the qth row contains $\bar{k}$ if any box of the pth row is unfilled.

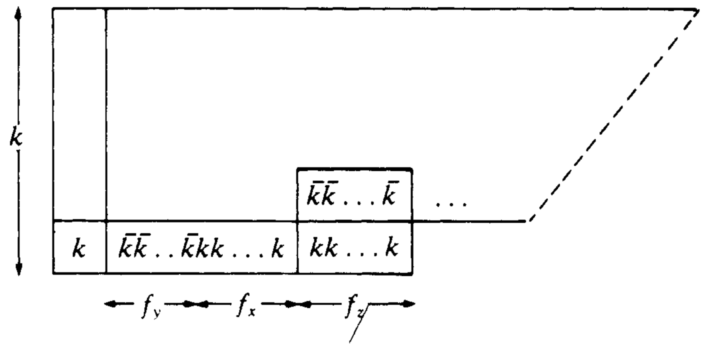
In the case of the restriction from Sp($2k$) to Sp($2k-2$)$\times$U(1) the only non-vanishing contributions to $\langle\lambda\rangle$ are those enumerated by entering k or $\bar{k}$ into the boxes of F^{λ} in such a way that if $p=k$ then each and every box of the kth row contains either k or $\bar{k}$.

Modification rules are not required in restricting from SO($2k+1$) to SO($2k$), but they are most certainly required in dealing with the most complicated case: that of restricting from SO($2k$) to SO($2k-2$)$\times$U(1). In the case of the characters $[\lambda]$ with

$p<k$ there is no problem, of course, since the modification rules are not called into play. It is convenient to treat the remaining characters $[\lambda]_\pm$ with $p=k$ and $[\Delta;\lambda]_\pm$ with $p\leqslant k$ by considering both reducible and difference characters defined by (2.2).

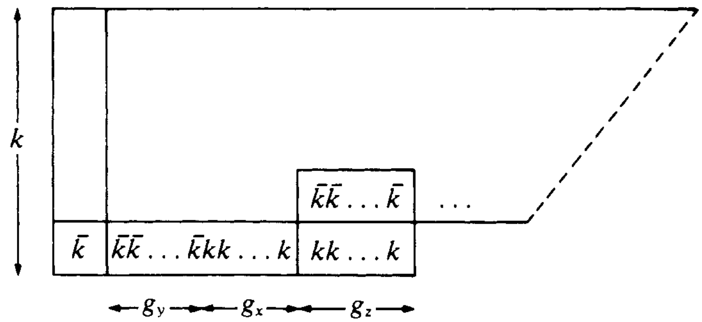
All contributions to the difference character $[\square;\lambda]''$ of SO(2k) are excluded by the modification rules for SO($2k-2$), except for those enumerated by entering k or $\bar{k}$ into the boxes of $F^{\square;\lambda}$ in such a way that if $p=k$ then each and every box of the kth row contains either k or $\bar{k}$. Moreover, pairs of terms corresponding to the diagrams

$D_1^{\square;\lambda}=$



and

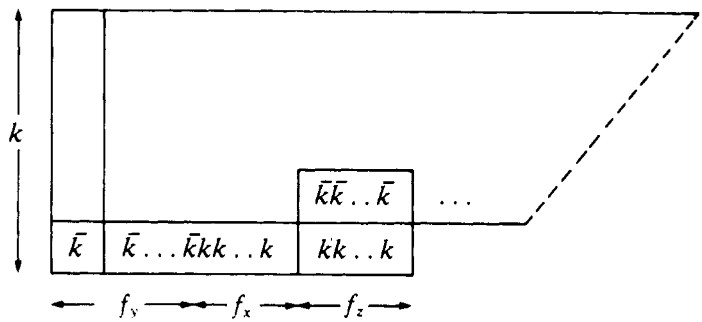
$D_2^{\square;\lambda}=$



cancel if $g_x=f_x+1\geqslant 1$, $g_y=f_y-1\geqslant 0$ and $g_z=f_z\geqslant 0$ by virtue of the fact that in such a case if the first, $D_1^{\square;\lambda}$, is associated with the U(1) character $\{x-y+1\}$ then the second, $D_2^{\square;\lambda}$, is necessarily associated with the character $-\{x-y-1\}$; the only terms which are not cancelled in this way are those corresponding to $D_1^{\square;\lambda}$ with $f_y=0$ or to $D_2^{\square;\lambda}$ with $g_x=0$. Such diagrams have entries filling all the boxes of the kth row to the left of any possible $\frac{\bar{k}}{k}$ pairs with entries which are either all k's or all $\bar{k}$'s.

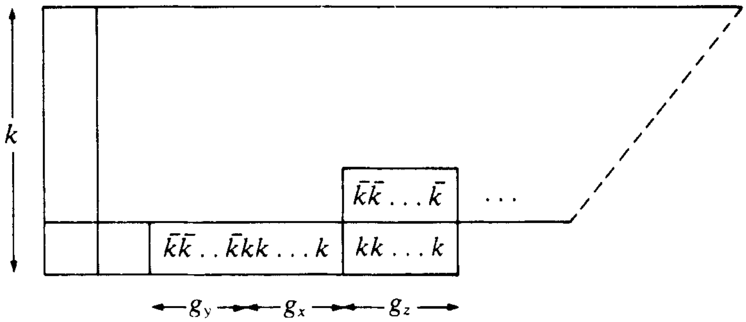
The corresponding contributions to the reducible character $[\square;\lambda]$ of SO(2k) are apparently more numerous since the modification rules imply that 0, 1 or 2 boxes may be left unfilled in the kth row of $F^{\square;\lambda}$. However, pairs of terms corresponding to the diagrams

$D_3^{\square;\lambda}=$



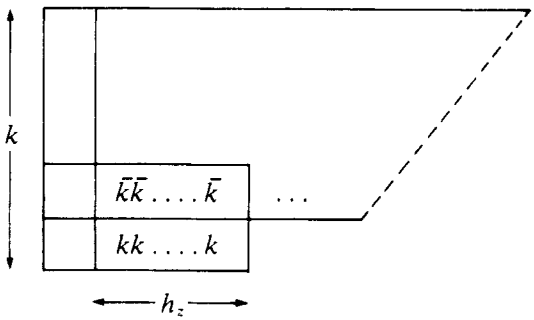
and

$D_4^{\square;\lambda} =$



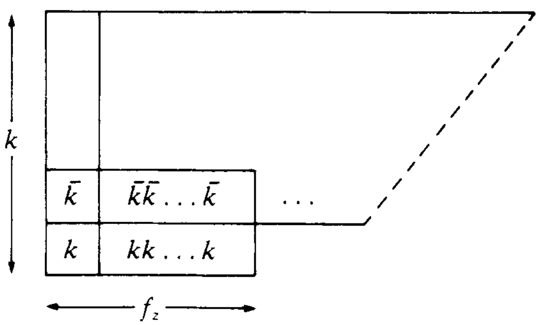
cancel if $g_x = f_x - 1 \geqslant 0$, $g_y = f_y - 1 \geqslant 0$ and $g_z = f_z \geqslant 0$ since in such a case both $D_3^{\square;\lambda}$ and $D_4^{\square;\lambda}$ are associated with the same U(1) character $\{x-y\}$ and the modification rule introduces a negative sign in deleting the two unfilled boxes of $D_4^{\square;\lambda}$. It follows that the only uncancelled terms are those of type $D_3^{\square;\lambda}$ with either $f_x = 0$ and $f_y \geqslant 1$, or $f_y = 0$ and $f_y \geqslant 1$, or $f_x = 0$ and $f_y = 0$. In the first two cases these diagrams coincide with those of type $D_2^{\square;\lambda}$ with $g_x = 0$ or $D_1^{\square;\lambda}$ with $f_y = 0$ respectively. In the third case a term associated with precisely the same U(1) character $\{x-y\}$ arises after modification of the diagram

$D_5^{\square;\lambda} =$



with $h_z = f_z - 1 \geqslant 0$, through the deletion of the two unfilled boxes at the foot of the first column. This modification is equivalent to replacing $D_5^{\square;\lambda}$ by

$D_6^{\square;\lambda} =$

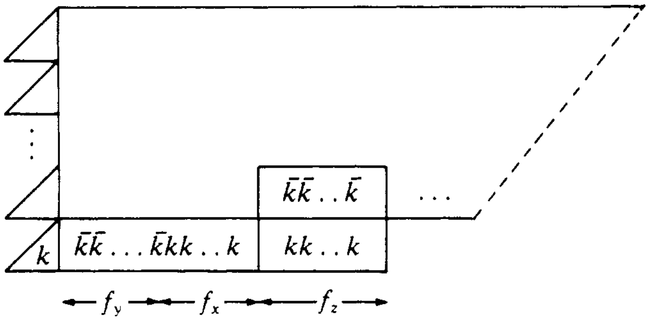


which is identical to $D_3^{\square;\lambda}$ with $f_x = f_y = 0$.

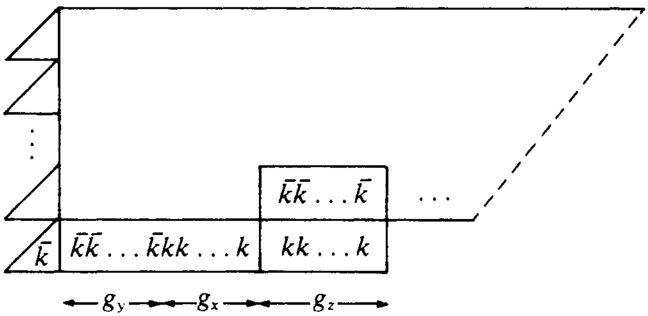
Finally, turning to the spin characters $[\Delta;\lambda]$ and $[\Delta;\lambda]'$ of SO$(2k)$, the modification rules imply that in both cases only 0 or 1 boxes may be left unfilled in the kth row of $F^{\Delta;\lambda}$.

There are thus four types of diagram to consider:

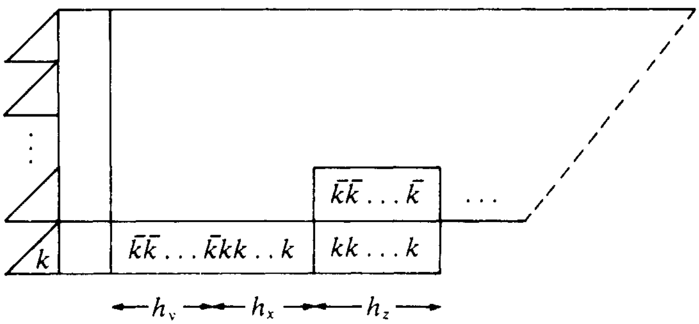
$D_1^{\Delta;\lambda} =$



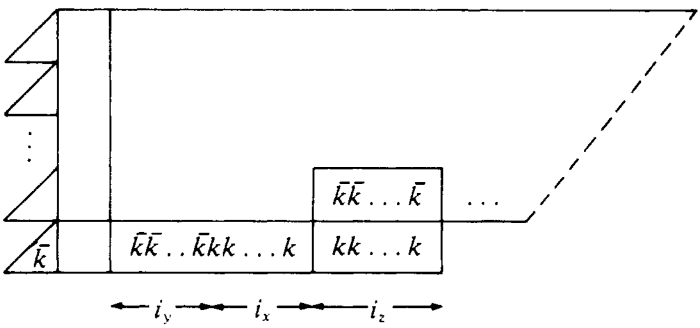
$D_2^{\Delta;\lambda} =$



$D_3^{\Delta;\lambda} =$



$D_4^{\Delta;\lambda} =$



In the case of the reducible character $[\Delta; \lambda]$ of SO$(2k)$ the modification rule, incorporating as it does a negative sign, produces a cancellation between terms corresponding to the diagrams $D_1^{\Delta;\lambda}$ and $D_4^{\Delta;\lambda}$ if $i_x = f_x \geqslant 0$, $i_y = f_y - 1 \geqslant 0$ and $i_z = f_z \geqslant 0$ and a cancellation between terms corresponding to $D_2^{\Delta;\lambda}$ and $D_3^{\Delta;\lambda}$ if $h_x = g_x - 1 \geqslant 0$, $h_y = g_y \geqslant 0$ and $h_z = g_z \geqslant 0$. Thus the only uncancelled contributions arise from $D_1^{\Delta;\lambda}$ with $f_y = 0$ and from $D_2^{\Delta;\lambda}$ with $g_x = 0$. Similar cancellations take place in the case of contributions to the difference character $[\Delta; \lambda]''$ of SO($2k$). The only change is that the negative sign producing the cancellation comes not from the modification rule but from the U(1) character $-\{x - y - \frac{1}{2}\}$ rather than $+\{x - y + \frac{1}{2}\}$. Thus the uncancelled contributions from $D_1^{\Delta;\lambda}$ and $D_2^{\Delta;\lambda}$ are positive and negative respectively. They involve diagrams in which each of the boxes in the kth row to the left of any $\frac{\bar{k}}{k}$ pairs is filled with k or $\bar{k}$ according as the half box at the foot of the first column is filled with k or $\bar{k}$, respectively.

This completes the task of describing a combinatorial algorithm based on Young diagrams for enumerating all the terms arising as a result of using the branching rules of table 3.

5. Standard Young tableaux

The enumeration procedure described in § 4 involves inserting entries k, $\bar{k}$ and 0 into certain of the boxes, dotted boxes and half boxes of various Young diagrams F^{λ}, $F^{\bar{\mu};\lambda}$ and $F^{\Delta;\lambda}$. The unfilled boxes, dotted boxes and half boxes constitute Young diagrams F^{σ}, $F^{\bar{\tau};\sigma}$ and $F^{\Delta;\sigma}$. Repeating the procedure using entries $k-1$ and $\overline{k-1}$, then again using $k-2$ and $\overline{k-2}$, and so on until finally using the entries 1 and $\bar{1}$, leads to the construction of various arrays T^{λ}, $T^{\bar{\mu};\lambda}$ and $T^{\Delta;\lambda}$ known as standard Young tableaux. These are arrays in which every box, dotted box and half box is filled by some entry.

The entries in these standard Young tableaux are taken from one or more of the sets

$$S_k = \{1, 2, \ldots, k\} \qquad S_{\bar{k}} = \{\bar{1}, \bar{2}, \ldots, \bar{k}\} \qquad \text{and } S_0 = \{0\}$$

and the distribution of the entries in the tableaux is governed by various combinations of rules $R_1, R_2, \ldots, R_8$. With respect to the total ordering

$$\bar{1} < 1 < \bar{2} < 2 < \ldots < \bar{k} < k < 0 \tag{5.1}$$

these rules take the form:

R_1: the entries in boxes are non-decreasing from left to right across each row;

R_2: the entries in dotted boxes are non-decreasing from right to left across each row;

R_3: the entries in boxes and in dotted boxes are strictly increasing from top to bottom down each column;

R_4: if the lowest rows containing entries i and $\bar{i}$ are $r(i)$ and $r(\bar{i})$ respectively then $r(i) + r(\bar{i}) \leqslant i$ for $i = 1, 2, \ldots, k$;

R_5: if the lowest rows containing entries i and $\bar{i}$ are $r(i)$ and $r(\bar{i})$ respectively then $r(i) \leqslant i$ and $r(\bar{i}) \leqslant i$ for $i = 1, 2, \ldots, k$;

R_6: if the lowest row containing an entry 0 is $r(0)$ then $r(0) \leqslant k$;

R_7: the entry in the half box of the ith row is either i or $\bar{i}$ for $i = 1, 2, \ldots, k$;

R_8: no entry i may appear to the right of an entry $\bar{i}$ in the ith row unless it also lies immediately below an entry $\bar{i}$ for $i = 1, 2, \ldots, k$.

The subset of rules appropriate to a particular set of standard Young tableaux depends not only upon the type of tableaux T^{λ}, $T^{\bar{\mu};\lambda}$ or $T^{\Delta;\lambda}$ but also upon the group under consideration. It is convenient to denote the standard Young tableaux of type T^{λ} by A^{λ}, B^{λ}, C^{λ} or D^{λ} for SU(k), SO($2k+1$), Sp($2k$) or SO($2k$) respectively, those of type $T^{\bar{\mu};\lambda}$ by $U^{\bar{\mu};\lambda}$ for U(k), and those of type $T^{\Delta;\lambda}$ by $B^{\Delta;\lambda}$ or $D^{\Delta;\lambda}$ for SO($2k+1$) or SO($2k$) respectively. With this notation the relevant set of rules for each of the classical groups is specified in table 4.

Table 4. Rules governing the entries in standard Young tableaux.

Group	Tableau	Entries	Rules
U(k)	$U^{\bar{\mu};\lambda}$	$1, 2, 3, \ldots, k$ in F^{λ}; $\bar{1}, \bar{2}, \ldots \bar{k}$ in $F^{\bar{\mu}}$	R_1, R_2, R_3, R_4
SU(k)	A^{λ}	$1, 2, \ldots, k$ in F^{λ}	R_1, R_3
SO($2k+1$)	B^{λ}	$\bar{1}, 1, \bar{2}, 2, \ldots, \bar{k}, k, 0$ in F^{λ}	R_1, R_3, R_5, R_6, R_8
	$B^{\Delta;\lambda}$	$\bar{1}, 1, \bar{2}, 2, \ldots \bar{k}, k, 0$ in F^{λ}; $\bar{1}, 1, \bar{2}, 2, \ldots \bar{k}, k$ in F^{Δ}	$R_1, R_3, R_5, R_6, R_7, R_8$
Sp($2k$)	C^{λ}	$\bar{1}, 1, \bar{2}, 2, \ldots \bar{k}, k$ in F^{λ}	R_1, R_3, R_5
SO($2k$)	D^{λ}, $D^{\Delta;\lambda}$	$\bar{1}, 1, \bar{2}, 2, \ldots \bar{k}, k$ in F^{λ}; $\bar{1}, 1, \bar{2}, 2, \ldots \bar{k}, k$ in F^{Δ}	R_1, R_3, R_5, R_7, R_8.

In the case $k=5$ the various sets of standard Young tableaux are exemplified by:

$$U^{(\overline{21^2};31)} \quad \begin{array}{ccccc} \bar{3} & \bar{1} & 3 & 5 & 5 \\ & \bar{2} & 5 & & \\ & \bar{5} & & & \end{array}$$

$$A^{(531^2)} \quad \begin{array}{ccccc} 1 & 3 & 3 & 5 & 5 \\ 2 & 4 & 5 & & \\ 4 & & & & \\ 5 & & & & \end{array}$$

$$B^{(531^2)} \quad \begin{array}{ccccc} \bar{1} & 2 & \bar{3} & 0 & 0 \\ \bar{3} & 3 & 0 & & \\ 5 & & & & \\ 0 & & & & \end{array} \qquad B^{(\Delta;3^21)} \quad \begin{array}{cccccc} \bar{1} & / & \bar{2} & \bar{2} & 5 \\ 2 & / & 2 & 4 & 0 \\ \bar{3} & / & 0 & & \\ \bar{4} & / & & & \\ 5 & / & & & \end{array}$$

$$C^{(531^2)} \quad \begin{array}{ccccc} \bar{1} & 1 & 2 & 2 & 4 \\ \bar{2} & 2 & 3 & & \\ \bar{4} & & & & \\ 4 & & & & \end{array}$$

$$D^{(531^2)} \quad \begin{array}{ccccc} \bar{1} & \bar{2} & 2 & 2 & 4 \\ 2 & 2 & 3 & & \\ \bar{4} & & & & \\ 4 & & & & \end{array} \qquad D^{(\Delta;3^21)} \quad \begin{array}{cccccc} \bar{1} & / & \bar{1} & \bar{4} & \bar{4} \\ 2 & / & \bar{3} & 4 & 5 \\ \bar{3} & / & 3 & & \\ \bar{4} & / & & & \\ 5 & / & & & \end{array}$$

where the symbol / separates entries in half boxes from those in boxes and, for typographical convenience, the boxes themselves have not been drawn.

In dealing with the groups SO($2k+1$) and SO($2k$) it is necessary to introduce some further notational devices.

The entries in the first column of boxes of T^{λ} or the column of half boxes of $T^{\Delta;\lambda}$ constitute a vector $\boldsymbol{l}=(l_1, l_2, \ldots, l_k)$, where l_i is the leading entry in the ith row of T^{λ}

or $T^{\Delta;\lambda}$ for $i=1, 2, \ldots, k$. In the case of T^λ with λ such that $p<k$ it is to be understood that $l_i=0$ for $i=p+1, p+2, \ldots, k$. It is then convenient to define a signature vector, $\boldsymbol{\varepsilon}=(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$, associated with each standard Young tableau T^λ or $T^{\Delta;\lambda}$ such that

$$\varepsilon_i = \begin{cases} 1 & \text{if } l_i = 1 \text{ and } l_{i-1} \neq \bar{i} \\ -1 & \text{if } l_i = \bar{i} \\ \pm 1 & \text{if } l_i = i \text{ and } l_{i-1} = \bar{i} \\ 0 & \text{if } l_i \neq i \text{ or } \bar{i} \end{cases} \tag{5.2}$$

for $i=1, 2, \ldots, k$. The signature, ε, of each standard Young tableau T^λ or $T^{\Delta;\lambda}$ is in turn defined by

$$\varepsilon = \prod_{i=1}^{k} \varepsilon_i. \tag{5.3}$$

The rule R_7 is such that for each standard Young tableau $T^{\Delta;\lambda}$ the signature vector $\boldsymbol{\varepsilon}$ is unique. This is not the case for each standard Young tableau T^λ although the signature itself, ε, is unique. In general the number of distinct signature vectors $\boldsymbol{\varepsilon}$ associated with each standard Young tableau T^λ is 2^β, where the duplication parameter β is defined by

$$\beta = \sum_{i=2}^{k} \delta_{l_i,i}\, \delta_{l_{i-1},\bar{i}}. \tag{5.4}$$

Thus β is the number of pairs of leading entries $\bar{i}$ and i in the first boxes of the $(i-1)$th and ith rows, respectively, of T^λ.

6. Weight multiplicities

Just as the Young diagrams of § 4 provide a means of enumerating all the terms arising from each irreducible representation λ_G on restriction from each classical group G to a subgroup $H \times \mathrm{U}(1)$, so the standard Young tableaux of § 5 provide a means of enumerating all the terms arising on restriction from G to $\mathrm{U}(1) \times \mathrm{U}(1) \times \ldots \times \mathrm{U}(1)$. This is a consequence of their construction being determined by the branching and modification rules appropriate to the group–subgroup chains:

$$\mathrm{U}(k)\!\downarrow\!\mathrm{U}(k-1)\times\mathrm{U}(1)\!\downarrow\!\mathrm{U}(k-2)\times\mathrm{U}(1)\times\mathrm{U}(1)\!\downarrow\!\ldots\!\downarrow\!\mathrm{U}(1)\times\mathrm{U}(1)\times\ldots\times\mathrm{U}(1) \tag{6.1}$$

$$\mathrm{SO}(2k+1)\!\downarrow\!\mathrm{SO}(2k)\!\downarrow\!\mathrm{SO}(2k-2)\times\mathrm{U}(1)\!\downarrow\!\mathrm{SO}(2k-4) \times\mathrm{U}(1)\times\mathrm{U}(1)\!\downarrow\!\ldots\!\downarrow\!\mathrm{U}(1)\times\mathrm{U}(1)\times\ldots\times\mathrm{U}(1) \tag{6.2}$$

$$\mathrm{Sp}(2k)\!\downarrow\!\mathrm{Sp}(2k-2)\times\mathrm{U}(1)\!\downarrow\!\mathrm{Sp}(2k-4)\times\mathrm{U}(1)\times\mathrm{U}(1)\!\downarrow\!\ldots\!\downarrow\!\mathrm{U}(1)\times\mathrm{U}(1)\times\ldots\times\mathrm{U}(1) \tag{6.3}$$

$$\mathrm{SO}(2k)\!\downarrow\!\mathrm{SO}(2k-2)\times\mathrm{U}(1)\!\downarrow\!\mathrm{SO}(2k-4)\times\mathrm{U}(1)\times\mathrm{U}(1)\!\downarrow\!\ldots\!\downarrow\!\mathrm{U}(1)\times\mathrm{U}(1)\times\ldots\times\mathrm{U}(1). \tag{6.4}$$

Each standard Young tableau determines a unique weight vector $\boldsymbol{w}=(w_1, w_2, \ldots, w_k)$ such that

$$w_i = n_i - n_{\bar{i}} \qquad \text{for } i=1, 2, \ldots, k \tag{6.5}$$

where n_i is the sum of half the number of entries i in half boxes and the number of entries i in boxes of the tableau, whilst $n_{\bar{i}}$ is the same quantity defined for $\bar{i}$ rather than i. The entries 0 make no contribution to $\boldsymbol{w}$.

For example, in the case of the standard Young tableaux displayed in § 5 the corresponding weight vectors $\boldsymbol{w}$ are given by

$U^{(\overline{21^2};31)}$	$\boldsymbol{w}=(\bar{1}\ \bar{1}\ 0\ 0\ 2)$		
$A^{(531^2)}$	$\boldsymbol{w}=(1\ 1\ 2\ 2\ 4)$		
$B^{(531^2)}$	$\boldsymbol{w}=(\bar{1}\ 1\ \bar{1}\ 0\ 1)$	$B^{(\Delta;3^21)}$	$\boldsymbol{w}=(\bar{\tfrac{1}{2}}\ \bar{\tfrac{1}{2}}\ \bar{\tfrac{1}{2}}\ \tfrac{1}{2}\ \tfrac{3}{2})$
$C^{(531^2)}$	$\boldsymbol{w}=(0\ 2\ 1\ 1\ 0)$		
$D^{(531^2)}$	$\boldsymbol{w}=(\bar{1}\ 3\ 1\ 1\ 0)$	$D^{(\Delta;3^21)}$	$\boldsymbol{w}=(\bar{\tfrac{3}{2}}\ \tfrac{1}{2}\ \bar{\tfrac{1}{2}}\ \bar{\tfrac{3}{2}}\ \tfrac{3}{2})$

where, of course, $w_i=\bar{n}$ implies $w_i=-n$.

The use of the terminology 'weight vector' for $\boldsymbol{w}$ is justified by the fact that if a standard Young tableau T^{λ_G} associated with an irreducible representation λ_G has weight vector $\boldsymbol{w}$ then the restriction of λ_G from G to $\mathrm{U}(1)\times\mathrm{U}(1)\times\ldots\times\mathrm{U}(1)$ gives rise to at least one term of the form $\{w_1\}\times\{w_2\}\times\ldots\times\{w_k\}$. The ith component w_i is associated with the character $\{w_i\}=\exp(\mathrm{i}w_i\phi_i)$ of the ith group U(1) for $i=1,2,\ldots,k$. It follows that

$$\chi^{\lambda_G}(\boldsymbol{\phi})=\sum_{T^{\lambda_G}} 2^{\beta_G}\exp\mathrm{i}(\boldsymbol{w}\cdot\boldsymbol{\phi}) \tag{6.6}$$

where the summation is carried out over all the standard Young tableaux T^{λ_G} associated with λ_G, $\boldsymbol{w}$ is the weight vector of T^{λ_G} and β_G the corresponding duplication parameter.

The problem of evaluating the weight multiplicities $M_{\boldsymbol{w}}^{\lambda_G}$ of (2.1) is thus reduced to little more than that of enumerating the appropriate standard Young tableaux whose weight vectors are $\boldsymbol{w}$. It is only necessary to incorporate the duplication factor when dealing with the standard Young tableaux B^{λ} and D^{λ} of SO($2k+1$) and SO($2k$) respectively.

The results are made explicit in table 5 where

$$\delta_{\boldsymbol{t},\boldsymbol{w}}=\begin{cases}1 & \text{if } \boldsymbol{t}=\boldsymbol{w}\\ 0 & \text{if } \boldsymbol{t}\neq\boldsymbol{w}\end{cases} \tag{6.7}$$

and $\boldsymbol{u}$, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ are the weight vectors of the standard Young tableaux $U^{\bar{\mu};\lambda}$, A^{λ}, B^{λ} or $B^{\Delta;\lambda}$, C^{λ}, and D^{λ} or $D^{\Delta;\lambda}$ respectively.

It should be noted that in the case of SO($2k$) the signature ε is needed to determine whether D^{λ} contributes to $[\Delta;\lambda]_+$ or $[\Delta;\lambda]_-$ for $p\leq k$.

The duplication factors owe their origin to the duplication of diagrams of the type $D_6^{\square;\lambda}$ arising from both $D_3^{\square;\lambda}$ with $f_x=f_y=0$ and $D_5^{\square;\lambda}$, as explained in § 3. With the definition (5.4) this factor is 2^{β} for SO($2k+1$) and is $2^{\beta-1}$ for SO($2k$), by virtue of (2.2) which implies that

$$[\Delta;\lambda]_{\pm}=\tfrac{1}{2}([\Delta;\lambda]\pm[\Delta;\lambda]'') \qquad [\square;\lambda]_{\pm}=\tfrac{1}{2}([\square;\lambda]\pm[\square;\lambda]''). \tag{6.8a, b}$$

The very straightforward manner in which the explicit formulae of table 5 may be used to determine weight multiplicities is illustrated by the examples shown in table 6.

One or two points should be made concerning these results. Firstly there is of course a link between both the standard Young tableaux and the weight multiplicities

Table 5. Weight multiplicities.

G	$M_w^{\lambda_G}$	
U(k)	$M_w^{\{\bar\mu;\lambda\}} = \sum_{U^{\bar\mu;\lambda}} \delta_{u,w}$	for $p+q \leq k$
SU(k)	$M_w^{\{\lambda\}} = \sum_{A^\lambda} \delta_{a,w}$	for $p \leq k-1$
SO($2k+1$)	$M_w^{[\lambda]} = \sum_{B^\lambda} 2^\beta \delta_{b,w}$	for $p \leq k$
	$M_w^{[\Delta;\lambda]} = \sum_{B^{\Delta;\lambda}} \delta_{b,w}$	for $p \leq k$
Sp($2k$)	$M_w^{\langle\lambda\rangle} = \sum_{C^\lambda} \delta_{c,w}$	for $p \leq k$
SO($2k$)	$M_w^{[\lambda]} = \sum_{D^\lambda} 2^\beta \delta_{d,w}$	for $p \leq k-1$
	$M_w^{[\lambda]_\pm} = \sum_{D^\lambda} \delta_{d,w}(2^{\beta-1}\delta_{\varepsilon,0}+\delta_{\varepsilon,\pm 1})$	for $p=k$
	$M_w^{[\Delta;\lambda]_\pm} = \sum_{D^{\Delta;\lambda}} \delta_{d,w}\delta_{\varepsilon,\pm 1}$	for $p \leq k$

of SU(k) and U(k). This comes about because the restriction of the irreducible representation $\{\bar\mu;\lambda\}$ of U(k) to SU(k) yields the irreducible representation $\{\rho\}$ for which F^ρ is formed from $F^{\bar\mu;\lambda}$ by replacing each column of dotted boxes of $F^{\bar\mu;\lambda}$ of length $\tilde\mu_j$ by a column of boxes of F^ρ of length $\tilde\rho_j = \tilde\mu_j$ for $j = 1, 2, \ldots, \mu_1$. The remaining columns of boxes of F^ρ coincide with those of $F^{\bar\mu;\lambda}$ so that $\tilde\rho_j = \tilde\lambda_{j-\mu_1}$ for $j = \mu_1+1, \mu_1+2, \ldots, \mu_1+\lambda_1$ (King 1970). In terms of standard Young tableaux this produces a one-to-one correspondence between tableaux $T^{\bar\mu;\lambda}$ and T^ρ with each column of negative entries of $T^{\bar\mu;\lambda}$ replaced by its complement in the set $\{1, 2, \ldots, k\}$. Thus as indicated in table 6 the columns

$$\begin{matrix}\bar 2\\ \bar 3\\ \bar 5\end{matrix} \qquad \text{and} \qquad \begin{matrix}\bar 3\\ \bar 4\\ \bar 5\end{matrix}$$

correspond to the columns $\begin{smallmatrix}1\\4\end{smallmatrix}$ and $\begin{smallmatrix}1\\2\end{smallmatrix}$ under the restriction from U(5) to SU(5). This is a direct consequence of the identities

$$\exp \mathrm{i}(-\phi_2-\phi_3-\phi_5) = \exp \mathrm{i}(\phi_1+\phi_4)$$

and

$$\exp \mathrm{i}(-\phi_3-\phi_4-\phi_5) = \exp \mathrm{i}(\phi_1+\phi_2)$$

which follow from the constraint

$$\exp \mathrm{i}(\phi_1+\phi_2+\ldots+\phi_k) = 1 \tag{6.9}$$

appropriate to SU(k), with $k = 5$. It is thus no accident that

$$M_{(10\bar 1 1\bar 1)}^{\{\bar 1^3;21\}} = M_{(21020)}^{\{32\}}.$$

Table 6. Calculation of weight multiplicities.

G	λ_G	w		T^{λ_G}	$M_w^{\lambda_G}$
U(5)	$\{\overline{1^3};\ 2\ 1\}$	$(1\ 0\ \bar{1}\ 1\ \bar{1})$		$\bar{2}$ 1 2 ; $\bar{3}$ 1 4 $\bar{3}$ 4 ; $\bar{4}$ 4 $\bar{5}$; $\bar{5}$	2
SU(5)	{3 2}	(2 1 0 2 0)		1 1 2 ; 1 1 4 4 4 ; 2 4	2
SO(7)	[3 2]	(0 1 2)		$\bar{2}$ 2 2 ; $\bar{2}$ 2 3 ; $\bar{2}$ $\bar{3}$ 3 ; 2 3 3 ; 2 3 0 3 3 ; 2 3 ; 3 3 ; 0 0 ; 3 0	6
			2^β	1 ; 2 ; 1 ; 1 ; 1	
	$[\Delta;\ 2\ 1^2]$	$(\frac{3}{2}\ \frac{3}{2}\ \frac{1}{2})$		1/1 2 ; 1/1 $\bar{3}$; 1/1 2 ; 1/1 3 ; 1/1 0 2/$\bar{3}$; 2/2 ; 2/3 ; 2/2 ; 2/2 3/3 ; 3/3 ; $\bar{3}$/0 ; $\bar{3}$/0 ; 3/0	5
Sp(6)	⟨3 2⟩	(0 1 2)		$\bar{1}$ 1 2 ; $\bar{1}$ 1 3 ; $\bar{2}$ 2 2 ; $\bar{2}$ 2 3 ; 2 $\bar{3}$ 3 3 3 ; 2 3 ; 3 3 ; 2 3 ; 3 3	5
SO(6)	[3 2]	(0 1 2)		$\bar{2}$ 2 2 ; $\bar{2}$ 2 3 ; 2 $\bar{3}$ 3 3 3 ; 2 3 ; 3 3	4
			2^β	1 ; 2 ; 1	
	$[2^2\ 1]_+$	$(1\ 1\ \bar{1})$		1 2 $\bar{3}$ $\bar{3}$ 3	1
			ε	0	
			$2^{\beta-1}$	1	
	$[2^2\ 1]_-$	$(1\ 1\ \bar{1})$		1 $\bar{2}$; 1 $\bar{3}$; 1 2 2 2 ; 2 3 ; $\bar{3}$ $\bar{3}$ $\bar{3}$; $\bar{3}$; 3	3
			ε	$\bar{1}$; $\bar{1}$; 0	
			$2^{\beta-1}$	— ; — ; 1	
	$[2^2\ 1]_+$	(1 0 0)		1 2 ; 1 $\bar{2}$; 1 $\bar{3}$ $\bar{2}$ 3 ; 2 $\bar{3}$; $\bar{3}$ 3 $\bar{3}$; 3 ; 3	3
			ε	1 ; 1 ; 0	
			$2^{\beta-1}$	— ; — ; 1	
	$[2^2\ 1]_-$	(1 0 0)		1 2 ; 1 $\bar{2}$; 1 $\bar{3}$ $\bar{2}$ 3 ; 2 3 ; $\bar{3}$ 3 3 ; $\bar{3}$; 3	3
			ε	$\bar{1}$; $\bar{1}$; 0	
			$2^{\beta-1}$	— ; — ; 1	
	$[\Delta;\ 2\ 1^2]_+$	$(\frac{3}{2}\ \frac{3}{2}\ \frac{1}{2})$		1/1 2 ; 1/1 $\bar{3}$ 2/$\bar{3}$; 2/2 3/3 ; 3/3	2
	$[\Delta;\ 2\ \ 1^2]_-$	$(\frac{3}{2}\ \frac{3}{2}\ \frac{1}{2})$		—	0

More generally

$$M_{\boldsymbol{w}}^{\{\bar{\mu};\lambda\}} = M_{\boldsymbol{w}+\boldsymbol{c}}^{\{\rho\}} \tag{6.10}$$

where the relationship between ρ and $\bar{\mu}$; λ is as given above and $\boldsymbol{c} = (\mu_1\mu_1 \ldots \mu_1)$.

Secondly, it is well known that each group character $\chi^{\lambda_G}(\boldsymbol{\phi}_G)$ is invariant under the action of the Weyl symmetry group W_G. This leads to the symmetry properties of weight diagrams which is made precise through the identity

$$M_{S\boldsymbol{w}}^{\lambda_G} = M_{\boldsymbol{w}}^{\lambda_G} \qquad \text{for each } S \in W_G. \tag{6.11}$$

For the classical groups G the Weyl symmetry groups W_G are well known. For both U(k) and SU(k) the Weyl symmetry group is the symmetric group S_k of all permutations of the components of $\boldsymbol{w}$. For both SO($2k+1$) and Sp($2k$) the Weyl symmetry group is the hyperoctahedral group Q_k of all permutations and independent sign changes of the components of $\boldsymbol{w}$, whilst for SO($2k$) the Weyl symmetry group is the subgroup of Q_k involving an even number of sign changes of the components of $\boldsymbol{w}$ (King and Al-Qubanchi 1981, Wybourne 1982).

For each weight $\boldsymbol{w}$ the set of all weights $\{S\boldsymbol{w}: S \in W_G\}$ is a set of G-equivalent weights all possessing the same weight multiplicity in any irreducible representation λ_G of G. Amongst this set there exists a unique dominant, highest weight $\boldsymbol{\omega}$ such that $\boldsymbol{\omega}$ is the highest weight of some irreducible representation ω_G of G. In calculating weight multiplicities it is thus only necessary to determine the dominant weight multiplicities $M_{\boldsymbol{\omega}}^{\lambda_G}$. It is convenient to adopt the notation of table 1 and (2.2) to replace the symbol $\boldsymbol{\omega}$ by the corresponding irreducible representation label (ω_G), where round brackets are now used in all cases to indicate that (ω_G) is really a weight label rather than a representation label. Thus in general

$$M_{\boldsymbol{w}}^{\lambda_G} = M_{(\omega_G)}^{\lambda_G} \tag{6.12}$$

where ω_G has highest weight vector $\boldsymbol{\omega} = S\boldsymbol{w}$ for some S in W_G. With this relation it follows from table 6 that

U(5)	$M_{(\bar{1}^2;1^2)}^{\{\bar{1}^3;21\}} = 2$	
SU(5)	$M_{(21)}^{\{32\}} = 2$	
SO(7)	$M_{(21)}^{[32]} = 6$	$M_{(\Delta;1^2)}^{[\Delta;21^2]} = 5$
Sp(6)	$M_{(21)}^{\langle 32\rangle} = 5$	
SO(6)	$M_{(21)}^{[32]} = 4$	
	$M_{(1^3)_-}^{[2^21]_+} = 1$	$M_{(1^3)_-}^{[2^21]_-} = 3$
	$M_{(1)}^{[2^21]_+} = 3$	$M_{(1)}^{[2^21]_-} = 3$
	$M_{(\Delta:1^2)_+}^{[\Delta;21^2]_-} = 2$	$M_{(\Delta:1^2)_+}^{[\Delta;21^2]} = 0.$

In the case of the group SO($2k$) there exists an outer automorphism such that

$$[\lambda]^{\dagger} = [\lambda] \qquad \text{for } p < k, \qquad [\lambda]_{\pm}^{\dagger} = [\lambda]_{\mp} \qquad \text{for } p = k$$

$$[\Delta;\lambda]_{\pm}^{\dagger} = [\Delta;\lambda]_{\mp} \qquad \text{for } p \leqslant k.$$

This operation merely changes the sign of the kth component of each weight vector. It follows that if λ and σ are partitions of l and s, respectively, into p and q parts, respectively, then for SO(2k)

$$M^{[\lambda]}_{(\sigma)_+} = M^{[\lambda]}_{(\sigma)_-} \qquad \text{for } p<k \text{ and } q=k \tag{6.13}$$

$$M^{[\lambda]_+}_{(\sigma)} = M^{[\lambda]_-}_{(\sigma)} \qquad \text{for } p=k \text{ and } q<k \tag{6.14}$$

$$M^{[\lambda]_+}_{(\sigma)_\pm} = M^{[\lambda]_-}_{(\sigma)_\mp} \qquad \text{for } p=k \text{ and } q=k \tag{6.15}$$

$$M^{[\Delta;\lambda]_+}_{(\Delta;\sigma)_\pm} = M^{[\Delta;\lambda]_-}_{(\Delta;\sigma)_\mp} \qquad \text{for } p\leqslant k \text{ and } q\leqslant k. \tag{6.16}$$

It follows from the previous results, therefore, that

$$\text{SO(6)} \qquad M^{[2^21]_+}_{(1^3)_+} = 3 \qquad M^{[2^21]_-}_{(1^3)_+} = 1$$

$$M^{[\Delta;21^2]_+}_{(\Delta;1^2)_-} = 0 \qquad M^{[\Delta;21^2]_-}_{(\Delta;1^2)_+} = 2.$$

The vanishing of the final weight multiplication $M^{[\Delta;21^2]_-}_{(\Delta;1^2)_+}$ in table 6 is no coincidence. Examination of the rules appropriate to the corresponding standard Young tableaux shows clearly that in general

$$M^{[\Delta;\lambda]_\pm}_{(\Delta;\sigma)_{\mp(-)^{l-s}}} = 0 \qquad \text{for } p\leqslant k \text{ and } q\leqslant k. \tag{6.17}$$

This leads directly to the result first given by Wybourne (1982):

$$M^{[\Delta;\lambda]_\pm}_{(\Delta;\sigma)_{\pm(-)^{l-s}}} = M^{[\Delta;\lambda]}_{(\Delta;\sigma)} \tag{6.18}$$

where use has been made of the notation (2.2), and

$$M^{[\Delta;\lambda]}_{(\Delta;\sigma)} = M^{[\Delta;\lambda]}_{(\Delta;\sigma)_+} = M^{[\Delta;\lambda]}_{(\Delta;\sigma)_-} \tag{6.19}$$

is the multiplicity of the weight $(\Delta;\sigma)$ in the irreducible representation $[\Delta;\lambda]$ of O(2k) (King and Plunkett 1976).

Similarly (6.14) implies that

$$M^{[\lambda]_+}_{(\sigma)} = M^{[\lambda]_-}_{(\sigma)} = \tfrac{1}{2}M^{[\lambda]}_{(\sigma)} \qquad \text{for } p\leqslant k \text{ and } q<k \tag{6.20}$$

where $M^{[\lambda]}_{(\sigma)}$ is again the multiplicity of the weight (σ) in the irreducible representation $[\lambda]$ of O(2k). Unfortunately no such simple result is valid for $M^{[\lambda]_+}_{(\sigma)_\pm}$ and $M^{[\lambda]_-}_{(\sigma)_\pm}$. Indeed our example indicates that $M^{[2^21]}_{(1^3)} = 4$ for the irreducible representation $[2^21]$ of O(6) and that these four weights are unequally distributed between the irreducible representations $[2^21]_+$ and $[2^21]_-$ of SO(6). The enumeration of standard Young tableaux provides a means of distributing the weights in accordance with the general result

$$M^{[\lambda]}_{(\sigma)} = M^{[\lambda]_+}_{(\sigma)_\pm} + M^{[\lambda]_-}_{(\sigma)_\pm} = M^{[\lambda]_\pm}_{(\sigma)_+} + M^{[\lambda]_\pm}_{(\sigma)_-} \qquad \text{for } p=q=k. \tag{6.21}$$

Using this technique the dominant weight multiplicities have been calculated for each irreducible representation $[\lambda]_+$ of SO(2k) with $p=k$ and $l\leqslant 6$. The results are given in table 7. The results may be checked by making use of the k-dependent formulae for $M^{[\lambda]}_{(\sigma)}$ (King and Plunkett 1976) for O(2k) and $M^{[\lambda]''}_{(\sigma)}$ (Wybourne 1982) for SO(2k).

It is also possible to use the standard Young tableaux to establish explicit k-dependent results. For example the standard Young tableaux:

$$\begin{matrix} \bar{1} & 3 & 4 \\ \bar{2} & i & \\ \bar{i} & & \end{matrix} \qquad \begin{matrix} \bar{1} & 3 & i \\ 2 & 4 & \\ \bar{i} & & \end{matrix} \qquad \text{for } 5\leqslant i\leqslant k$$

Table 7. Non-vanishing dominant weight multiplicities $M^{[\lambda]_+}_{(\sigma)}$ and $M^{[\lambda]_+}_{(\sigma)_\pm}$ of SO(2k).

$k=1$ $[\lambda]_+$ \ (σ)	$(6)_+$	$(5)_+$	$(4)_+$	$(3)_+$	$(2)_+$	$(1)_+$
$[6]_+$	1					
$[5]_+$		1				
$[4]_+$			1			
$[3]_+$				1		
$[2]_+$					1	
$[1]_+$						1

$k=2$ $[\lambda]_+$ \ (σ)	$(51)_+$	$(42)_+$	$(3^2)_+$	$(41)_+$	$(32)_+$	(4)	$(31)_+$	$(31)_-$	$(2^2)_+$	$(2^2)_-$	(3)	$(21)_+$	$(21)_-$	(2)	$(1^2)_+$	$(1^2)_-$	(1)	(0)
$[51]_+$	1	1	1			1	1	1	1	1				1	1	1		1
$[42]_+$		1	1				1		1					1	1	1		1
$[3^2]_+$			1						1						1			1
$[41]_+$				1	1						1	1	1				1	
$[32]_+$					1							1					1	
$[31]_+$							1		1					1	1	1		1
$[2^2]_+$									1						1			1
$[21]_+$												1					1	
$[1^2]_+$															1			1

$k=3$ $[\lambda]_+$ \ (σ)	$(41^2)_+$	$(321)_+$	$(2^3)_+$	$(31^2)_+$	$(2^21)_+$	(4)	(31)	(2^2)	$(21^2)_+$	$(21^2)_-$	(3)	(21)	$(1^3)_+$	$(1^3)_-$	(2)	(1^2)	(1)	(0)
$[41^2]_+$	1	1	1			1	2	2	3	3					4	5		6
$[321]_+$		1	2				1	2	4	2					4	6		8
$[2^3]_+$			1						1						1	1		1
$[31^2]_+$				1	1						1	2	3	3			4	
$[2^21]_+$					1							1	3	1			3	
$[21^2]_+$									1						1	2		3
$[1^3]_+$													1				1	

$k=4$ $[\lambda]_+$ \ (σ)	$(31^3)_+$	$(2^21^2)_+$	$(21^3)_+$	(31)	(2^2)	(21^2)	$(1^4)_+$	$(1^4)_-$	(3)	(21)	(1^3)	(2)	(1^2)	(1)	(0)
$[31^3]_+$	1	1		1	1	3	6	6				6	10		16
$[2^21^2]_+$		1			1	2	6	3				3	8		15
$[21^3]_+$			1							1	3			6	
$[1^4]_+$							1						1		3

$k=5$ $[\lambda]_+$ \ (σ)	$(21^4)_+$	$(1^5)_+$	(21^2)	(1^4)	(1^3)	(2)	(1^2)	(1)	(0)
$[21^4]_+$	1		1	4		3	9		20
$[1^5]_+$		1			1			1	

$k=6$ $[\lambda]_+$ \ (σ)	$(1^6)_+$	(1^4)	(1^2)	(0)
$[1^6]_+$	1	1	3	10

imply that for U(k) with $k \geqslant 5$

$$M^{\{\bar{1}^3;21\}}_{(\bar{1}^2;1^2)} = 2k - 8.$$

No contributions arise for $i < 5$ since such terms are excluded by rule R_4.

Similarly the standard Young tableaux

$$\begin{array}{lll} 1\ 1\ 2 \\ 0\ 0 \end{array} \qquad \begin{array}{lll} 1\ 1\ 0 \\ 2\ 0 \end{array} \qquad \begin{array}{lll} 1\ 1\ \bar{2} \\ 2\ 2 \end{array}$$

$$\begin{array}{l} 1\ 1\ 2 \\ \bar{i}\ i \end{array} \qquad \begin{array}{l} 1\ 1\ \bar{i} \\ 2\ i \end{array} \qquad \begin{array}{l} 1\ 1\ i \\ 2\ \bar{i} \end{array} \qquad \text{for } 3 \leqslant i \leqslant k$$

imply that for SO($2k+1$) with $k \geqslant 2$

$$M^{[32]}_{(21)} = 3k - 3.$$

Excluding the first two from this set and including

$$\begin{array}{l} 1\ 1\ 2 \\ \bar{2}\ 2 \end{array}$$

then gives for Sp($2k$), with $k \geqslant 2$,

$$M^{\langle 32\rangle}_{(21)} = 3k - 4$$

whilst just excluding the first two tableaux gives for SO($2k$) with $k \geqslant 3$

$$M^{[32]}_{(21)} = 3k - 5.$$

Furthermore the standard Young tableaux

$$\begin{array}{l} 1\ /\ 1\ 2\ 0 \\ 2\ /\ 0 \\ \vdots \\ k\ / \end{array} \qquad \begin{array}{l} 1\ /\ 1\ 0\ 0 \\ 2\ /\ 2 \\ \vdots \\ k\ / \end{array} \qquad \begin{array}{l} 1\ /\ 1\ 2\ 2 \\ \bar{2}\ /\ 0 \\ \vdots \\ k\ / \end{array}$$

$$\begin{array}{l} 1\ /\ 1\ 2\ 0 \\ 2\ /\ i \\ \vdots \\ \bar{i}\ / \\ \vdots \\ k\ / \end{array} \qquad \begin{array}{l} 1\ /\ 1\ 2\ i \\ 2\ /\ 0 \\ \vdots \\ \bar{i}\ / \\ \vdots \\ k\ / \end{array} \qquad \begin{array}{l} 1\ /\ 1\ i\ 0 \\ 2\ /\ 2 \\ \vdots \\ \bar{i}\ / \\ \vdots \\ k\ / \end{array} \qquad \begin{array}{l} 1\ /\ 1\ 2\ 2 \\ \bar{2}\ /\ i \\ \vdots \\ \bar{i}\ / \\ \vdots \\ k\ / \end{array} \qquad \text{for } 3 \leqslant i \leqslant k$$

$$\begin{array}{l} 1\ /\ 1\ 2\ i \\ 2\ /\ i \\ \vdots \\ k\ / \end{array} \qquad \begin{array}{l} 1\ /\ 1\ 2\ i \\ 2\ /\ \bar{i} \\ \vdots \\ k\ / \end{array} \qquad \begin{array}{l} 1\ /\ 1\ \bar{i}\ i \\ 2\ /\ 2 \\ \vdots \\ k\ / \end{array} \qquad \text{for } 3 \leqslant i \leqslant k$$

$$\begin{array}{l} 1\ /\ 1\ 2\ i \\ 2\ /\ j \\ \vdots \\ \bar{i}\ / \\ \vdots \\ \bar{j}\ / \\ \vdots \\ k\ / \end{array} \qquad \begin{array}{l} 1\ /\ 1\ 2\ j \\ 2\ /\ i \\ \vdots \\ \bar{i}\ / \\ \vdots \\ \bar{j}\ / \\ \vdots \\ k\ / \end{array} \qquad \begin{array}{l} 1\ /\ 1\ i\ j \\ 2\ /\ 2 \\ \vdots \\ \bar{i}\ / \\ \vdots \\ \bar{j}\ / \\ \vdots \\ k\ / \end{array} \qquad \text{for } 3 \leqslant i < j \leqslant k$$

imply that for SO($2k+1$) with $k \geqslant 3$

$$M^{[\Delta;31]}_{(\Delta;1^2)} = \tfrac{1}{2}(3k^2 - k - 2)$$

whilst dropping those tableaux with an entry 0 and taking into account the signature of the remaining tableaux gives for SO($2k$) with $k \geqslant 3$ the multiplicities

$$M^{[\Delta;31]_+}_{(\Delta;1^2)_+} = \tfrac{1}{2}(3k^2 - 7k + 4) \qquad M^{[\Delta;31]_-}_{(\Delta;1^2)_+} = 0.$$

Finally the standard Young tableaux

$$\begin{array}{ll} 1 & \bar{i} \\ 2 & i \\ \cdot & \\ \cdot & \\ k & \end{array} \qquad \text{for } 2 \leqslant i \leqslant k \qquad \text{with } \varepsilon = 1$$

$$\begin{array}{ll} 1 & i \\ \cdot & j \\ \cdot & \\ \not{i} & \\ \cdot & \\ \bar{j} & \\ j & \\ \cdot & \\ \cdot & \\ k & \end{array} \qquad \text{for } 2 \leqslant i < j \leqslant k \qquad \text{with } \varepsilon = 0 \qquad \text{and } 2^{\beta-1} = 1$$

where $\not{i}$ indicates the omission of an entry i, imply that for SO($2k$) with $k \geqslant 2$

$$M^{[\square;1^2]_+}_{(\square;0)_+} = \tfrac{1}{2}k(k-1) \qquad M^{[\square;1^2]_-}_{(\square;0)_+} = \tfrac{1}{2}(k-1)(k-2).$$

This method of calculating the k-dependence of weight multiplicities was in fact used to check the extensive tabulation given earlier (King and Plunkett 1976). The new results presented here are those pertaining to SO($2k$), rather than to O($2k$), for which the enumeration technique is an alternative to the procedure of Wybourne (1982).

7. Discussion

It has been demonstrated that standard Young tableaux, constructed in accordance with well defined sets of rules dependent upon the classical group G under consideration, serve to define weight vectors of each irreducible representation λ_G, and by their enumeration they also determine the weight multiplicities. Furthermore it has been shown both that the method applies to the most difficult family of groups SO($2k$), as well as to U(k), SU(k), SO($2k+1$) and Sp($2k$), and that the method yields explicit k-dependent weight multiplicity formulae.

In addition, however, the standard Young tableaux, along with their distinct signature vectors if appropriate, label all the basis states of each irreducible representation and it is demonstrated in a subsequent paper that the column structure of such tableaux provides a means of writing down generating functions for all irreducible characters of the classical groups.

Finally it should be pointed out that the standard Young tableaux also serve to define various symmetric functions of considerable combinatorial interest. Indeed one can simply define

$$s^{\lambda_G}(\boldsymbol{x}) = \chi^{\lambda_G}(\boldsymbol{\phi}) \tag{7.1}$$

with $x_j = \exp(\mathrm{i}\phi_j)$ for $j = 1, 2, \ldots, k$, to obtain symmetric functions of the k-indeterminates $x_1, x_2, \ldots, x_k$. These functions are multinomials, possibly involving half integer powers, in the indeterminates $x_1, x_2, \ldots, x_k$ and their inverses $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k$. They are symmetric in the sense that

$$s^{\lambda_G}(S\boldsymbol{x}) = s^{\lambda_G}(\boldsymbol{x}) \qquad \text{for each } S \in \mathrm{W}_G \tag{7.2}$$

where W_G is the Weyl symmetry group of G.

In the case of U(k) with $\lambda_G = \{\lambda\}$ this definition yields the very well known Schür functions (Littlewood 1940, p 84, Macdonald 1979, p 23)

$$s_\lambda = s^{\{\lambda\}}(\boldsymbol{x}). \tag{7.3}$$

For example the standard Young tableaux of U(3)

$$\begin{array}{llllllll} 1\;1 & 1\;2 & 1\;1 & 1\;2 & 2\;2 & 1\;3 & 1\;3 & 2\;3 \\ 2 & 2 & 3 & 3 & 3 & 2 & 3 & 3 \end{array}$$

serve to define

$$s^{\{21\}} = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + 2x_1x_2x_3$$

which is manifestly symmetric under permutations of x_1, x_2 and x_3.

In precisely the same way the standard Young tableaux of Sp(4)

$$\begin{array}{llllll} \bar{1}\;\bar{1} & \bar{1}\;1 & 1\;1 & \bar{1}\;\bar{1} & \bar{1}\;1 & 1\;1 \\ \bar{2} & \bar{2} & \bar{2} & 2 & 2 & 2 \end{array}$$

$$\begin{array}{llllllll} \bar{1}\;\bar{2} & \bar{1}\;\bar{2} & \bar{1}\;2 & \bar{1}\;2 & 1\;\bar{2} & 1\;\bar{2} & 1\;2 & 1\;2 \\ \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 2 \end{array}$$

$$\begin{array}{ll} \bar{2}\;\bar{2} & \bar{2}\;2 \\ 2 & 2 \end{array}$$

serve to define

$$s^{\langle 21\rangle} = \bar{x}_1^2\bar{x}_2 + x_1^2\bar{x}_2 + \bar{x}_1^2x_2 + x_1^2x_2 + \bar{x}_1\bar{x}_2^2 + x_1\bar{x}_2^2 + \bar{x}_1x_2^2 + x_1x_2^2 + 2(\bar{x}_1 + x_1 + \bar{x}_2 + x_2).$$

Similarly the two subsets of these appropriate to SO(4)

$$\begin{array}{llllllll} \bar{1}\;\bar{1} & \bar{1}\;\bar{2} & \bar{1}\;2 & 1\;1 & 1\;\bar{2} & 1\;2 & \bar{2}\;\bar{2} & \bar{2}\;2 \\ \bar{2} & \bar{2} & \bar{2} & 2 & 2 & 2 & 2 & 2 \end{array}$$

and

$$\begin{array}{llllllll} \bar{1}\;\bar{1} & \bar{1}\;\bar{2} & \bar{1}\;2 & 1\;1 & 1\;\bar{2} & 1\;2 & \bar{2}\;\bar{2} & \bar{2}\;2 \\ 2 & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 \end{array}$$

define

$$s^{[21]_+} = \bar{x}_1^2\bar{x}_2 + x_1^2x_2 + \bar{x}_1\bar{x}_2^2 + x_1x_2^2 + \bar{x}_1 + x_1 + \bar{x}_2 + x_2$$

and

$$s^{[21]_-} = \bar{x}_1^2x_2 + x_1^2\bar{x}_2 + \bar{x}_1x_2^2 + x_1\bar{x}_2^2 + \bar{x}_1 + x_1 + \bar{x}_2 + x_2.$$

Just as the monomial symmetric functions (Macdonald 1979, p 11)

$$m_{21}=x_1^2x_2+x_1x_2^2+x_1^2x_3+x_1x_3^2+x_2^2x_3+x_2x_3^2$$

and

$$m_{1^3}=x_1x_2x_3$$

provide a convenient homogeneous basis for s_{21} so

$$m^{[21]_+}=\bar{x}_1^2\bar{x}_2+x_1^2x_2+\bar{x}_1\bar{x}_2^2+x_1x_2^2$$
$$m^{[21]_-}=\bar{x}_1^2x_2+x_1^2\bar{x}_2+x_1\bar{x}_2^2+\bar{x}_1x_2^2$$

and

$$m^{[1]}=\bar{x}_1+x_1+\bar{x}_2+x_2$$

provide a homogeneous basis for $s^{\langle 21\rangle}$, $s^{[21]_+}$ and $s^{[21]_-}$.

Without pursuing the matter further here it should be clear that a treasure house of symmetric functions is available, generalising Schür functions and monomial symmetric functions.

References

Abramsky Y J and King R C 1970 *Nuovo Cimento* A **67** 153–216
Black G R E, King R C and Wybourne B G 1983 *J. Phys. A: Math. Gen.* **16** 1555–89
Black G R E and Wybourne B G 1983 *Preprint*, University of Canterbury
Boerner H 1970 *Representations of Groups* 2nd edn (Amsterdam: North-Holland)
Delaney R M and Gruber B 1969 *J. Math. Phys.* **10** 252–65
Gilmore R 1970a *J. Math. Phys.* **11** 513–23
—— 1970b *J. Math. Phys.* **11** 1853–4
Hegerfeldt G C 1967 *J. Math. Phys.* **8** 1195–6
King R C 1970 *J. Math. Phys.* **11** 280–94
—— 1971 *J. Math. Phys.* **12** 1588–98
—— 1975a *J. Phys. A: Math. Gen.* **8** 429–49
—— 1975b *Lecture Notes in Physics* (New York: Springer) **50** 490–9
—— 1981 *C.R. Math. Rep. Acad. Sci. Canada* **3** 149–53
—— 1982 *Physica* **114A** 345–9
King R C and Al-Qubanchi A H A 1981 *J. Phys. A: Math. Gen.* **14** 51–75
King R C, Luan Dehuai and Wybourne B G 1981 *J. Phys. A: Math. Gen.* **14** 2509–38
King R C and El-Sharkaway N G I 1982 *C.R. Math. Rep. Acad. Sci. Canada* **4** 299–304
King R C and Plunkett S P O 1976 *J. Phys. A: Math. Gen.* **9** 863–87
Littlewood D E 1940 *The Theory of Group Characters* (London: OUP)
Macdonald L G 1979 *Symmetric Functions and Hall Polynomials* (Oxford: Clarendon)
Miller W 1966 *Pacific J. Math.* **16** 341–6
Murnaghan F D 1938 *The Theory of Group Representations* (Baltimore: Johns Hopkins)
Newell M J 1951 *Proc. R. Irish Acad.* **54** 153–63
Patera J and Sharp R T 1979 *Lecture Notes in Physics* (New York: Springer) **94** 175–83
Stanley R P 1980 *J. Math. Phys.* **21** 2321–6
Weyl H 1931 *The Theory of Groups and Quantum Mechanics* (London: Methuen)
Whippman M L 1965 *J. Math. Phys.* **6** 1534–9
Wybourne B G 1982 *J. Phys. A: Math. Gen.* **15** 2687–97
Zhelobenko D P 1962 *Russian Math. Surveys* **17** 1–94